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H. FAST

THIN SETS IN CARTESIAN PRODUCTS CAN BE OPAQUE

2e boerhaavestraat 49 amsterdam

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Thin sets in Cartesian Products can be Opaque

by

H. Fast*)

ABSTRACT

In a cartesian product $X = P_n X_n$ of separable metric spaces X_n, each endowed with a countable class of Borel measures, there exists a Borel subset H which is "thin" in the sense of having all its factor-space projections 0-dimensional (topologically), Hausdorff 0-dimensional and also of measure zero for all the respective measures but still having the property that each image of a fixed complete separable metric space T into X by a continuous and reasonably regular mapping must meet H.
By imposing further restricting conditions on the spaces X_n as well as on the class of admissible mappings from T into X, one achieves that H is closed.

KEY WORDS & PHRASES: *Hausdorf measure, metric space, 0-dimensional space, Borel set.*

*) This work was done while the author was visiting the Mathematical Centre at Amsterdam, the Netherlands.

1. INTRODUCTION

The title suggests optical intuition which motivated the present paper. It originates with the question: can a subset of a space be very "thin" and yet be able to stop any "light-ray", i.e. to meet any curve of a certain family of curves in this space, a family which is quite "ample"? In the sequel we give precise meaning to the terms used here loosely, and we answer this question in the affirmative. At this point let us mention only this much: we restrict ourselves in this paper to a space of a particular kind: a countable product of metric spaces, each carrying a countable class of Borel measures. The "rays" are a class of injections into this space of another metric space: a parameter space. In the particular case when as the parameter space the real line is taken the injections are just curves, and the optical analogy becomes particularly close.

The precise statement of the results comes in page 4 of this paper.

2. PRELIMINARIES

Following a known method of Caratheodory, a finite non-negative real-valued function ψ defined on the set of all balls in a separable metric space generates an outer measure-function. After being restricted to Borel sets it gives rise to a non-negative Borel measure in the space. In the sequel this Borel measure will be called *the measure generated by* ψ and denoted m^{ψ}. As an example of such a function ψ may serve the function χ^{α}, $(\alpha > 0)$, defined for a ball $u(x,r)$ as: $\chi^{\alpha}[u(x,r)] = r^{\alpha}$ (here $u(x,r)$ stands for the ball with centre x and radius r). The measure $\Lambda^{\alpha} = m^{\chi^{\alpha}}$ thus generated is known as the *α-dimensional Hausdorff measure*. Recall here the concept of a *Hausdorff 0-dimensional* set: a set of Λ^{α}-measure zero for all $\alpha > 0$.

We shall make the following standing assumption about a function ψ which in the sequel will be referred to as *generating function:*

(2.1) $\inf\{\psi(u) : u \subset v\} = 0$

for any fixed ball v of the space. This assumption holds for instance for the generating function χ^{α}.

Let X_n, n=1,2,... be metric spaces. With each X_n let there be associated a system $\{\psi_{n,i}\}_{i=1}^{\infty}$ of generating functions. Let $X = P_n X_n$ be the cartesian product of the spaces X_n, endowed with the usual product-topology. A subset $A \subset X$ will be called $\{\psi_{n,i}\}_{n,i}$*-thin* or simply *thin* if the projections $\text{proj}_n A$, n = 1,2,... of A on the factor-spaces are:

(a) 0-dimensional (in the topological sense)

(b) Hausdorff 0-dimensional

(c) of $m^{\psi_{n,i}}$-measure zero for all n,i = 1,2,... .

A will be called *strongly* $\{\psi_{n,i}\}_{n,i}$*-thin* or simply *strongly thin* if in addition to (a)-(c) it satisfies the condition:

(d) $\text{proj}_n A$, n = 1,2,... are closed.

The following immediate consequences may be noted:

(1) *All the sets* $\text{proj}_n A$ *are nowhere-dense in their respective spaces* X_n *when A is strongly thin.*

(2) *A closure of a strongly thin set remains a strongly thin set.*
Setting $m^{[\psi_{n,i}]_n}$ for the product-measure $P_n m^{\psi_{n,i}}$ generated on Borel sets of X by the factor-space measures $m^{\psi_{n,i}}$, one obtains:

(3) *A thin set A is 0-dimensional in X and of* $m^{[\psi_{n,i}]_n}$*-measure zero for* $i = i,2,...$

(4) *The class of thin sets is countably-additive.*

Indeed, (1) is an immediate consequence of (a) and (d). (2) follows from the fact that in view of (d) taking closure of A does not change the $\text{proj}_n A$ which are closed already. Regarding (3), the 0-dimensionality of A follows from (a) [if each point of $\text{proj}_n A$ has arbitrarily small open -and-closed neighbourhoods then A has the same property with respect to product-neighbourhoods at each of its points], and the last fact is a trivial consequence of the fact that $m^{[\psi_{n,i}]_n} A = 0$ as soon as $m^{\psi_{n,i}}\ \text{proj}_n A = 0$ for even one value of n. (4) is clear.

It would be convenient to introduce at this point a number of terms and concepts which we shall find useful both in the statement of the results as well as in the subsequent proof.

Let $\mathbb{A}$ and $\mathbb{B}$ be two classes of subsets of the same set. $\mathbb{A}$ is said to be *dense* in $\mathbb{B}$ if for any non-empty set $B \in \mathbb{B}$ there is a non-empty set $A \in \mathbb{A}$ such that $A \subset B$. $\mathbb{A}$ is said to be a *refinement* of $\mathbb{B}$ if for any set $A \in \mathbb{A}$ there is a set $B \in \mathbb{B}$ such that $A \subset B$. $\mathbb{A}$ is said to be a *dense refinement* of $\mathbb{B}$ if it is both dense in $\mathbb{B}$ as well as its refinement. In the case of subsets of a metric (topological) space we define in addition: $\mathbb{A}$ is *strongly dense* in $\mathbb{B}$ and $\mathbb{A}$ is a *strong refinement* of $\mathbb{B}$ and finally, $\mathbb{A}$ is a *strongly dense refinement* of $\mathbb{B}$ if the requirement $A \subset B$ is strenghtened to $\bar{A} \subset B$. Clearly, if $\mathbb{A}$ is dense (resp. strongly dense) in $\mathbb{B}$ then there is a subclass of $\mathbb{A}$ which is a dense refinement (resp. a strongly dense refinement) of $\mathbb{B}$.

Let T and Z be metric spaces. Let $C(Z,T)$ denote the class of continuous mappings $g: T \to Z$. It is easy to see that $C(P_n X_n, T) = P_n C(X_n, T)$. Ball(Z) denotes the class of balls in the space Z. For $v \in \text{Ball}(Z)$, cntr(v) is the centre of v. A mapping $g \in C(Z,T)$ is called *open* at a point t_0 if for any $v \in \text{Balls}(T)$ such that $t_0 = \text{cntr}(v)$, $g(t_0) \in \text{Int}\, g(v)$; it is called open on a set if it is open at each point of this set. $C^0(Z,T)$ denotes the class of mappings from T into Z, continuous and open on T.

A mapping $g \in C(PX_n, T)$ is called *coordinate-open* (at a point, on a set) if all its factor-space projections: $\text{proj}_n \circ g$, n=1,2,... are open (relative to the respective spaces X_n).

A class of balls in a metric space with equal radii is called *equi-radial*. A class of mappings $G \subset C^0(Z,T)$ is *equi-open* if for any equi-radial class $U \subset \text{Ball}(T)$ there is an equi-radial class $U' \subset \text{Ball}(Z)$ which is dense in the class $G[U] = \{g(u) : g \in G, u \in U\}$. A class $G \subset C(Z,T)$ is *equi-continuous* if for any equi-radial class $U \subset \text{Ball}(Z)$ there is an equi-radial class $U' \subset \text{Ball}(T)$ such that $\text{cntr}U' = \{\text{cntr}(u') : u' \in U'\} \subset G^{-1}[\text{cntr}(U)] = \{g^{-1}(\text{cntr}(u)) : g \in G, u \in U\}$, and that U' is dense in $G^{-1}[U]$.

A class $G \subset C(P_n X_n, T)$ is *coordinate equi-open* (resp. *coordinate equi-continuous*) if all the classes $G_n = \{\text{proj}_n\, g : g \in G\}$, n = 1,2,... are equi-open (resp. equi-continuous).

A mapping $g \in C(PX_n, T)$ with all the projections $\text{proj}_n \circ g$ open on some open subset of T will be called a *general mapping*. Evidently, coordinate -open mappings on T are general mappings.

3. MAIN RESULTS

We state the results of this paper in the form of the following:

THEOREM: *Let* X_n, $n = 1,2,\ldots$ *and* T *be metric spaces. Let on each* X_n *there be given a countable class* $\{\psi_{n,i}\}_{i=1}^{\infty}$ *of generating functions. Under the assumptions:*

(a_1): *each* X_n, $n=1,2,\ldots$, *is separable,*

(a_2): T *is complete,*

(a_3): T *is separable,*

there exists in $X = P_n X_n$ *a* $\{\psi_{n,i}\}_{n,i}$*-thin Borel subset H with the property for any general mapping* $g : T \to X$

$$(3.1) \qquad H \cap g(T) \neq \emptyset \ .$$

If (a_1), (a_2) *and, moreover*

(a_4): *each* X_n *has the property that its closed balls are compact,*

then given any coordinate equi-open and coordinate equi-continuous class $G \subset C(X,T)$ *there exists a strongly* $\{\psi_{n,i}\}_{n,i}$*-thin closed subset* H *of* X *with the property that (3.1) holds for any* $g \in G$.

A question which the author considers to be of interest but which remains unanswered in the present paper is:

Does the first part of the result hold if instead of a general mapping we use a continuous mapping with the property that all the $proj_n \circ g$, $n = 1,2,\ldots$ *are open simultaneously on an uncountable set?*

The next few paragraphs contain the proof of the theorem.

4. PRELIMINARY LEMMAS ON NETS

We define yet one more general concept: Let $\mathbb{A}$ be a class of subsets of a metric space Z. An at most countable subclass E of Ball(Z) which is a strongly dense refinement of $\mathbb{A}$ will be called an *($\mathbb{A}$)-net*.

In the case when all the balls constituting a net are mutually disjoint, the net will be called a *disjoint net.*

REMARK: A strongly dense refinement of a countable set is automatically a net.

Let Z and T be two *separable* metric spaces. Let $V \subset \mathrm{Ball}(T)$ and $F \subset C(Z,T)$.

LEMMA 4.1. *Let* $F \subset C^0(Z,T)$. *Then* F[V] *has a disjoint net. Under the assumptions that* F *is equi-open and* V *is equi-radial, the net may be assumed to be equi-radial and have elements whose distances are bounded away from zero.*

PROOF. Since by our assumption Int $f(v) \neq \emptyset$ for $(f,v) \in F \times V$, there exists a class $U \subset \mathrm{Ball}(z)$ which is strongly dense in F[V]. Let D be a dense countable subset of the set cntr(U), $D = \{d_n\}_n$. Let $U' = \{u'_n\}_n$ be the subclass of U with $\mathrm{cntr}(u'_n) = d_n$, n=1,2,... . Select from U' a disjoint subclass $U'' = \{u''_k\}_k$, $u''_k = u'_{n_k}$, k=1,2,... setting: $u''_1 = u'_1$ and inductively, taking as n_k the smallest natural for which

$$\mathrm{cntr}(u'_{n_k}) \notin \bigcup_{i=1}^{k-1} u'_{n_i} .$$

If we set D = cntr(U'), then clearly

$$D = \mathrm{cntr}(U') \subset \bigcup_k u''_k ,$$

hence also $\mathrm{cntr}(U) \subset \bar{D} \subset \bigcup_k u''_k$. This means that for any $u \in U$ there is an u''_k such that $\mathrm{cntr}(u) \in u''_k$. In particular this implies the existence of a ball $u'''_{u,k}$ with $u'''_{u,k} \subset u \cap u''_k$. The class $\{u'''_{u,k} : u \in U, k=1,2,...\}$ is a strong refinement of both classes U and U" and dense in U hence it is a strong dense refinement of F[V] and disjoint, thus a disjoint F[V]-net.

Under the stronger assumptions (F equi-open and V equi-radial) the class U above may be taken as equi-radial. Here and in the sequel we shall find useful the following notation: if $u \in \mathrm{Ball}(Z)$ and $\alpha > 0$ then ${}^{\alpha}u$ is the ball concentric with u with radius α times the radius of u. In addition, we write: ${}^{\alpha}U = \{{}^{\alpha}u : u \in U\}$. Form U' and U" as above and consider the classes

${}^{3/4}U$, ${}^{3/4}U'$, ${}^{3/4}U''$. Since ${}^{3/4}U''$ is disjoint, the balls from ${}^{1/4}U''$ are at positive mutual distances bounded away from zero. Given u we have that for some k

$$\mathrm{cntr}({}^{3/4}u) \in {}^{3/4}u''_k,$$

hence

$$\overline{{}^{1/5}u''_k} \subset u.$$

This means that ${}^{1/5}U''$ is strongly dense in U. Since $U'' \subset U' \subset U$, it follows that U" is also a refinement of U. Thus, ${}^{1/5}U''$ is an (F[V])-net which meets the stronger requirements. □

LEMMA 4.2. *Let* $V \subset \mathrm{Ball}(T)$, V *at most countable,* $F \subset C^0(Z,T)$, E *an* $(F[{}^{\frac{1}{2}}V])$-*net. There exists a strongly dense refinement* V' *of* V *which is simultaneously a refinement of* $F^{-1}[E]$. *If, moreover,* V *is equi-radial,* F *is equi-open and equi-continuous then* V' *may be assumed equi-radial as well.*

PROOF. By the definition of an $(F[{}^{\frac{1}{2}}V])$-net there exists a selection

$$(f,v) \to e_{f,v} \in E \ ; \ \bar{e}_{f,v} \subset f({}^{\frac{1}{2}}v)$$

acting from $F \times V$ into E. Since $\mathrm{cntr}(e_{f,v}) \in f({}^{\frac{1}{2}}v)$, we have $f^{-1}(\mathrm{cntr}(e_{f,v})) \cap ({}^{\frac{1}{2}}v) \neq \emptyset$, thus there exists a selection

$$(f,v) \to t_{f,v} \in f^{-1}(\mathrm{cntr}(e_{f,v})) \cap ({}^{\frac{1}{2}}v)$$

from $F \times V$ into T and also a selection

$$(f,v) \to v'_{f,v} \in \mathrm{Ball}(T) \ : \ \overline{v'_{f,v}} \subset f^{-1}(e_{f,v}) \cap v.$$

The class $V' = \{v'_{f,v} : (f,v) \in F \times V\}$ is a strongly dense refinement of V and also a refinement of $F^{-1}[E]$. Since V is at most countable by assumption this class V' is a (V)-net. Under the stronger assumptions, lemma 4.1 yields us an equi-radial E. By equi-continuity of F, the selection of $v'_{f,v}$ may be equi-radial as well. □

LEMMA 4.3. *Let* ϕ_i, $1 \le i \le k$, *be generating functions on* Z. *For any* $\varepsilon > 0$ *there exists a disjoint* (F[V])*-net* E *such that*

$$(4.1) \qquad \sum\{\phi_i(u) : u \in E\} < \varepsilon \qquad \textit{for } 1 \le i \le k.$$

PROOF. A dense refinement of a disjoint (F[V])-net is again a disjoint (F[V])-net. It suffices to note that due to the property (2.1) of a generating function, one may select in a given net a dense ball-refinement which satisfies one of the inequalities (4.1). By consecutive application of this for i=1,...,k, one obtains such a net, as claimed. □

5.

Let Z_n, n=1,2,... be *separable* metric spaces, T a metric space. Let $F_n \subset C^0(Z_n,T)$, n=1,2,... .

LEMMA 5.1. *There exist two sequences of classes:* $E_n \subset \mathrm{Ball}(Z_n)$ *and* $V_n \subset \mathrm{Ball}(T)$ *such that:*

(a) E_n *is a disjoint* $(F_n[\frac{1}{2}V])$*-net with balls of diameters less than 1/n.*
(b) V_{n+1} *is a* (V_n)*-net and also a refinement of* $F_n^{-1}[E_n]$.

Moreover, assuming that the F_n *are equi-open and equi-continuous,* E_n *and* V_n *may be assumed to be equi-radial; n=1,2,... .*

PROOF. Let a ball $v \in \mathrm{Ball}(T)$ be chosen arbitrarily. Set $V_1 = \{v\}$. Define both the sequences inductively: assuming that we have already E_k for $1 \le k \le n-1$ and V_k for $1 \le k \le n$, choose as E_n an $(F_n[\frac{1}{2}V_n])$-net and as V_{n+1} a (V_n)-net which is also a refinement of $F_n^{-1}[E_n]$ as established by the lemmas 4.1 and 4.2 (the condition on diameters of balls from E_n may be satisfied trivially). □

LEMMA 5.2. *Any* $f = [f_n]_n \in P_n F_n$ *determines two sequences:* $e_n = e_n(f) \in E_n$ *and* $v_n = v_n(f) \in V_n$, $n = 1,2,\ldots$ *such that:*

(5.1) $$\bar{e}_n \subset f_n(\tfrac{1}{2}v_n) \text{ and } \bar{v}_{n+1} \subset f^{-1}(e_n) \cap v_n.$$

PROOF. Take as v_1 the sole element of V_1. Define both the sequences inductively: assuming that we have already the e_k for $1 \le k \le n-1$ and v_k for $1 \le k \le n$, select e_n as an element from E_n satisfying the first of the inclusions (5.1) and then select as v_{n+1} an element from V_{n+1} satisfying the second (namely, taking the selected element v'_{f_n,v_n} used in the proof of lemma 4.2). □

LEMMA 5.3. *Assuming that* T *is complete we have*

(5.2) $$\bigcap_n f_n^{-1}(\cup E_n) \neq \emptyset$$

for any $f = [f_n]_n \in P_n F_n$.

[$\cup E_n$ stands for $\cup\{e : e \in E_n\}$].

PROOF. From the second inclusion in (5.1) we have $\bar{v}_{n+1} \subset v_n$ and $\bar{v}_{n+1} \subset f_n^{-1}(e_n)$ hence $v_{n+1} \subset f_n^{-1}(\cup E_n)$ for $n=1,2,\ldots$. Due to the condition that the balls of E_n are of radius smaller than $1/n$ we have by completeness of T:

$$\bigcap_n f_n^{-1}(\cup E_n) \supset \bigcap_n \bar{v}_n \neq \emptyset. \quad □$$

LEMMA 5.4. *Let for some infinite subset* N *of the naturals* (Z_n,F_n) *be independent of* $n \in N$, *say*

(5.3) $$(Z_n,F_n) = (Z,F) \qquad \textit{for all } n \in N.$$

Then the set

(5.4) $$\bigcap\{\cup E_n : n \in N\}$$

is 0*-dimensional in* Z. *If, moreover,* F_n*,* $n=1,2,\ldots$ *are equi-open and equi-continuous then the same holds for the set*

$$(5.5) \qquad \bigcap\{\overline{UE_n} : n \in N\}.$$

PROOF. For an arbitrary point $x \in \bigcap \{UE_n : n \in N\}$ the component $c_n(x)$ of x in UE_n is a ball of diameter less than $1/n$ and it is an open neighbourhood of x relatively closed in UE_n; it is, therefore, relatively closed in the resulting intersection. Thus the set is 0-dimensional. Under the strengthened assumptions and for appropriate nets E_n the set $\overline{UE_n}$ has $\overline{c_n(x)}$ as the component of x and the conclusion with respect to the set (5.5) is the same as above. □

LEMMA 5.5. *Let* $[\phi_{n,i}]_{i=1}^{\infty}$ *be a sequence of generating functions on* Z_n, $n=1,2,\ldots$. *Let for some infinite subset* N *of the naturals*

$$(5.6) \qquad (Z_n, F_n, \phi_{n,i}) = (Z, F, \phi_i) \quad \textit{for } n \in N$$

[*independence of* n]. *Then*

$$(5.7) \qquad m^{\phi_i}(\bigcap\{UE_n : n \in N\}) = 0 \quad \textit{for } i=1,2,\ldots .$$

Assuming, moreover, that the F_n *are equi-open and equi-continuous and that*

(5.8) $\qquad Z_n$ *have the property that their closed balls are compact*

we have

$$(5.9) \qquad m^{\phi_i}(\bigcap\{\overline{UE_n} : n \in N\}) = 0.$$

PROOF. By the lemma 4.3 the nets E_n may be assumed to satisfy $\Sigma\{\phi_{n,i}(u) : u \in E_n\} < 1/n$ for $1 \le i \le n$, $n=1,2,\ldots$ and in particular

$$\sum\{\phi_i(u) : u \in E_n\} < 1/n \quad \text{for } 1 \le i \le n, \quad n \in N.$$

Each E_n, $n \in N$ is a covering of the set (5.4) by balls with diameters smaller than 1/n. This implies (5.7).

Go over now to the strenghtened version. Let $z_n^0 \in Z_n$ an arbitrary point-selection with the condition:

(5.10) $\quad z_n^0 = z_{n'}^0$, whenever $Z_n = Z_{n'}$.

We have evidently the following:

$$Z = \cup\{u(z_n^0,n) \ : \ n \in N\}.$$

Due to our strengthened assumptions the E_n are equi-radial and specifically due to (5.8) the subclasses E_n^- of E_n, where

$$E_n^- = \{u \in E_n \ : \ u \subset u(z_n^0,n)\},$$

are all finite. Therefore, E_n may be assumed to satisfy

$$\sum\{\phi_{n,i}(u) \ : \ u \in E_n^-\} < 1/n \qquad \text{for } 1 \le i \le n, \quad n=1,2,\ldots$$

Indeed, if necessary, using (2.1) a sub-net may be selected in E_n, again an equi-radial one, for which it holds already. For a fixed $m \in N$ and for $n \in N$ large enough, E_n is a ball-covering of the portion $\cap\{\overline{\cup E_n} \ : \ n \in N\} \cap$ $\cap\, u(z_m^0,m)$ and by the same argument as above, the value of $m^{\phi_i^n}$ on this portion for all i=1,2,... is zero. This proves (5.9). □

6. PROOF OF THE MAIN THEOREM

Let $n \to [\kappa(n),\kappa'(n)]$ be a one-to-one mapping of the set of naturals onto the cartesian square of this set. Let X_n, n=1,2,... be *separable* metric spaces; T a *separable* and *complete* metric space. Let on each X_n there be given a sequence $[\psi_{n,i}]_{i=1}^{\infty}$ of generating functions. Set:

$$Z_n = X_{\kappa(n)}, \quad \phi_{n,i} = \psi_{\kappa(n),i}$$

Clearly, all the properties of the spaces X_n and the classes of mappings $g: T \to X_n$ are reflected in the same properties of the spaces Z_n and the mappings $f: T \to Z_n$. Note in particular, that making a selection $x_n^0 \in X_n$: : $x_n^0 = x_{n'}^0$, whenever $X_n = X_{n'}$, by taking $z_n^0 = x_{\kappa(n)}^0$ we obtain a sequence satisfying (5.10).

Let $g = [g_n]_n \in P_n X_n$ and let $f_n = g_{\kappa(n)}$. We have: $f_n = g_k$ for $\kappa(n) = k$ and

$$\bigcap_n f_n^{-1}(UE_n) = \bigcap_k \bigcap\{f_n^{-1}(UE_n) : \kappa(n) = k\} = \bigcap_k g_k^{-1}(\bigcap\{UE_n : \kappa(n) = k\}).$$

Setting $H_k = \bigcap\{UE_n : \kappa(n) = k\}$ we obtain the result of the lemma 5.3 in the form

$$(6.1) \qquad \bigcap_n g_n^{-1}(H_n) \neq \emptyset$$

for any $g = [g_n]_n \in P_n C^0(X_n,T)$. Since we have

$$g^{-1}(P_n H_n) = \{t \in T : g_n(t) \in H_n \text{ for } n=1,2,\ldots\} = \bigcap_n g_n^{-1}(H_n),$$

(6.1) takes the form

$$g^{-1}(P_n H_n) \neq \emptyset.$$

Or, setting $H = P_n H_n$, the form

$$(6.2) \qquad H \cap g(T) \neq \emptyset.$$

The set H is a Borel set in $X = P_n X_n$ and namely of G_δ type (because so were the sets H_n in their respective spaces). By lemma 5.5 (the weak version) we have $m^{\psi_{k,i}} H_k = 0$ for $k,i=1,2,\ldots$ (let us recall here that $\phi_{n,i} = \psi_{k,i}$ for $\kappa(n) = k$). The stronger versions of the lemmas 5.1, 5.4 and 5.5 assert that given equi-open and equi-continuous classes $G_n \subset C^0(X_n,T)$ the sets H_k (depending upon the collection of those classes) may be assumed, moreover, to be closed. In the latter version a mapping g is supposed to be taken from the product $P_n G_n$. Given any class G of mappings $g:T \to X = P_n X_n$, we can enclose it in the product: $G \subset P_n G_n$ where $G_n = \{proj_n \circ g : g \in G\}$. Thus without loss of generality we may assume that G has the form of such a product.

Thus, in order to show that the set H is thin in the sense exposed in page 2 in the weaker version and strongly thin in the stronger version only one detail is still missing: namely, that the sets H_n are Hausdorff 0-dimensional. But this is easy to achieve: without any loss of generality we may assume that for instance all the even i-indexed measures are Hausdorff measures: $m^{\psi_{n,2s}} = \Lambda^{1/s}$ (by taking $\psi_{n,2s} = \chi^{1/s}$). Hence Hausdorff 0-dimensionality follows directly.

Thus far the weaker version has been obtained only for mappings g from $P_n C^0(X_n,T)$. Let us extend this result just a little. Note that each of the balls $u \in Ball(T)$ after closure may be considered as a new parameter-space. Relativizing everything to the new parameter space $\bar{u}$ and using the notation $H_k(u)$ for the sets introduced earlier (in which the dependence upon the parameter-space is made explicit) we shall write $H(u) = P_k H_k(u)$ and, choosing a countable base $\mathcal{B}$ of T consisting of balls,

$$H = \cup\{H(u) : u \in \mathcal{B}\}$$

(note that this is the only place where we made use of the assumption of separability of T). By the property (4) page 2 (countable additivity of thin sets) H is a $\{\psi_{n,i}\}$-thin set. But for this set (6.1) holds for any general mapping $g:T \to X$. Indeed, for a suitable $\underline{u^0}$ from the countable base of balls in T the restriction $g_{|u^0}$ is in $P_n C^0(X_n,\overline{u^0})$. This concludes the proof of the theorem.

7. EXAMPLES AND APPLICATIONS

(1) Let m be a fixed natural and let the spaces X_n, n=1,2,... as well as T be identical with the Euclidean space R^m. Let G_n (again independent of n) be the class of translations of R^m (onto itself). Thus, as a matter of fact, G_n is identical with R^m again. Take $m^{\psi_{n,i}}$ just arbitrarily. Our result

yields in this case (we use the crucial relation in the form (6.1) rather than (6.2)): There are closed nowhere dense sets $H_n \subset R^m$, n=1,2,... each of Hausdorff dimension zero such that for any sequence of vectors (translations) $g_n \in R^m$, n=1,2,... there is a point $x \in R^m$ such that $x \in \bigcap_n g_n^{-1}H_n = \bigcap_n (H_n - g_n)$, i.e.

$$x + \{g_n\}_n = \{x + g_n\}_n \subset \bigcup_n H_n .$$

Since the set $\{g_n\}_n \subset R^m$ was arbitrary, this means: *There exists in* R^m *a Borel set of* F_σ *-type, of first category and of Hausdorff dimension zero with the property that any countable set of points in* R^m *may be placed within this set under a suitable translation.*

Before passing over to the next example let us point out that all our considerations in this paper are valid for an at most countable number of spaces X_n. In the forthcoming example it will be a finite number.

(2) Let X_n, $1 \le n \le m$ all be identical with $R^1 = R$ in which case we have: $P_n X_n = R^m$. Let T = R as well. Take the measures again arbitrarily. Then $P_n C^0(X_n, T) = PC^0(R,R)$ is the class of continuous mappings from R into R^m with the property that each coordinate-axis projection of such a mapping (being real-valued) has no extrema. Consider a Jordan curve in R^m together with all its possible locally-supporting hyperplanes perpendicular to the coordinate axis. If the set of support points is not dense on the curve, then any parametrization of the curve results in a general mapping. There exists in R^m a thin subset which each such a curve must meet. Restricting ourselves just to smooth Jordan curves it is easy to state a sufficient condition for the above condition: the tangent of such a curve must nowhere be parallel to one of the coordinate-hyperplanes or at least the set of points at which this occurs must not be dense on the curve.

And now the stronger version: Consider in R^m the class of Jordan curves which are smooth and parametrized by the entire R. Assume that for the curves of this class the direction of the tangent remains within a cone with axis the diagonal $\{x = [x_k]_{k=1}^m : x_1 = x_2 = \dots = x_m\}$ and of an angle $\alpha, 0 < \alpha < \pi/2$. It is easily seen that the class of corresponding parametrical mappings into R^m is coordinate equi-open and coordinate equi-continuous. Therefore, there exists a strongly thin set in R^m (depending on α) which meets each of the curves of the family.

(3) In specifying even more the example (2), take $m = 2$ and as the class of the curves (the stronger version) take the straight lines parallel to the main diagonal in R^2: $\{(x_1,x_2): x_1-x_2 = \xi\}$, $\xi \in R$. Our result takes up the form

$$\{x_1-x_2 : x_1 \in H_1, x_2 \in H_2\} = R$$

or in words: *There are two closed, Hausdorff* 0-*dimensional (and certainly nowhere-dense) sets on the line for which the set of distance between couples of points taken from those sets fills up the entire real line.* (c.f. [1]).

(4) Let $f: x \to [f_q]_{q=1}^{m-1}$, $x = [x_p]_{p-1}^{m}$ be a continuously differentiable mapping from R^m *onto* R^{m-1}. Since the (m-1)-minors of the Jacobi matrix $||\partial f_q/\partial x_p||_{p=1,\ q=1}^{m\quad m}$ are the coordinates of a tangent vector to the manifold $f^{-1}(f(x))$ at a point $x \in R^m$ at which they are all non-vanishing, any parametric representation $x = g(t)$ of $f^{-1}(f(x))$ (a piece about x) is coordinate-open at $t = g^{-1}(x)$. Therefore, the condition that they do not vanish anywhere in R^m is sufficient for the existence of a thin set H in R^m meeting any level set $f^{-1}(y)$, $y \in R^{m-1}$ or, in another form, being mapped by f onto R^{m-1}. Such a set H would be universal for all the mappings f from R^m onto R^{m-1} for which the said non-vanishing condition holds. Further examples and generalizations are yet possible.

REFERENCES

[1] EGGLESTON, H.G., *Note on certain n-dimensional sets*. Fund. Math. Vol. 36, (1949) pp.40-43.